The focus of this chapter is on key frame animation, motion pathways, and easing. Motion pathways exist in the geometric domain; easing exists in the temporal domain. It begins with a rather formal review of action and associated concepts such as speed, velocity, and acceleration, which form the physical basis for traditional easings. The discussion continues with a presentation of motion pathways and an examination of several ways that they may be created. Next, key frame techniques are explored in detail, both in a classic "cartoon" sense, as well as a computerized implementation in which key frames are defined with parameters.

The chapter concludes with an inventory of easing techniques, which, although not necessarily always natural, are extremely useful. It is possible to design easings that have a variety of changes. For example, it is possible to speed up and slow down or to construct an ease that accelerates for a while, maintains a constant velocity, accelerates again quickly, breaks quickly, slows down, darts forward—in general, to produce a tremendous amount of liveliness of movement, and yet be very smooth. This kind of creative movement, coupled with engaging spline pathways, can produce action that is very dramatic.

Keep in mind that animation is an art form of movement, and movement is a key element in character. The manner in which a camera moves compels viewers to respond—it brings them into or out of spaces quickly, it provides them with exhilaration when moving throughout small spaces quickly, or it instills them with a sense of peace. The manner in which the camera moves is how the

Some Basic Definitions
Motion Pathways
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Key Frame Animation
Easing
audience members themselves move, since the camera is their eye into the animated world.

Some Basic Definitions

Before beginning the discussion of key frame animation and easing, let us first define a few concepts that present a basic foundation for how objects move.

Action

Action refers to any and all kinds of change that occurs over time, including the change of position of objects, changes in an object's shape, changes in the position of the camera, and changes in the position of lights. Action is also created by changes in an object's geometry, by the brightness of lights, by rotation of objects and cameras, by surface patterns that are changing, by changes in the focus and focal length of the lens, and by special effects such as fades and wipes. Action must always occur at certain rates if they are to appear natural.

The rotation of a cube about its local y axis is an example of an action. The action can be separated into an action type, for example, the local y rotation, and an action parameter, in this example, an angular value. The most basic of action types are scaling, translation, and rotation. Action parameters depend upon the action type—for example, for rotation, the parameter may be an angle specified in degrees, while for translation, the parameters might be x, y, and z distances. Producing a sequence of animated images requires computing values for each action parameter for each frame and displaying the resulting images. Computers facilitate the computation of the action parameter values.

Vectors

A vector is a quantity specified by a size (or magnitude) and a direction (see Figure 10.1). There are certain things that can only be fully represented with the use of a vector, for example, a force. When a force is exerted, it is exerted in a particular direction and with a particular magnitude. Often, vectors are expressed or represented with the use of an arrow.
Any vector representation can be combined, or composited, with another vector. For example, a vector that represents force \( A \) can be combined with a vector that represents force \( B \) to create one resultant vector (see Figure 10.2). In the case of a resultant force, the result is a single force that is equivalent to the force that would be produced if the original two forces were applied. Forces can augment each other, as well as cancel each other out. Mathematically, resultant vectors are simple to calculate.

Vector is a term that is often used to refer to lines, and vector graphics has come to mean line drawings. Strictly speaking, each line is an instance of a vector because it can be represented with an initial position and a bearing (or directionality) and a magnitude that represents the same information as would two endpoints. We say this only to address any confusion the reader might have about our definition of the term here, which is slightly different than the line vector definition; it is also a definition that we intend to apply to a different purpose—in particular, the definitions of velocity and forces.

**Position, Motion, Speed, Velocity, and Acceleration**

**Position, Motion, and Speed** A position is a 2D or 3D location in space, expressed as \( xy \) or \( xyz \), respectively (see Figure 10.3). A displacement is a series of positions; it is not a single static, instantaneous position, but a set of positions that an object may take.

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**Figure 10.2** Left: A resultant vector. The resultant vector created from the 10-unit vector at 270 degrees and the 15-unit vector at 210 degrees is a 21.5-unit vector at 240 degrees.

**Figure 10.3** Above: A position is a 2D or 3D location in space, indicated in the figure by a point.
Figure 10.4 Above: A displacement is a series of positions. The positions may be on a straight line or a curve. The displacement illustrated here exists in the curve of a circle and goes around the y-axis.

Figure 10.5 Right: Speed is the rate at which an object changes position—for example, a truck might travel 15 miles/hour. Velocity is the rate of motion in a particular direction; it is a vector quantity and involves speed and direction.

(see Figure 10.4). Motion is a continuous change of position (although in the digital domain, it is usually approximated discretely). When in motion, an object changes its position at a particular rate, referred to as speed (see Figure 10.5). Therefore, speed indicates how fast an object is moving. Speed is the rate of motion, a measure of distance across time, and is expressed in units such as miles/hour, feet/second, knots (nautical miles/hour), or mach (speed relative to the speed of sound). In more general terms, speed indicates how fast an action is occurring—for example, how fast lights are dimming.

**Velocity** Velocity is the rate of motion in a particular direction (see again Figure 10.5). That is, velocity is defined as the speed at which an object moves, together with the direction in which it is moving. Therefore, velocity is a vector quantity; it has a magnitude and a direction. Because velocity is speed and direction, it is therefore expressed in 2D terms such as "60 miles/hour northwest"—a two-element vector of speed and compass heading (e.g., 60 miles/hour, 330 degrees). In 3D terms, velocity may be expressed as a four-element vector of speed and a direction normal. Uniform velocity occurs when velocity is constant; that is, when an object moves the same distance in the same direction at the same rate (see Figure 10.6). A graph of uniform velocity distance versus time produces a straight line.
### Acceleration

In life, an object such as an automobile at rest begins to move slowly and gains speed, or accelerates, before reaching a constant, uniform speed. The physical explanation is that it is necessary to overcome the inertia of the resting object. Conversely, objects decelerate and slow to a stop. **Acceleration** is the rate of change of velocity over time (see Figure 10.7). A drag racer accelerates quickly; a blimp accelerates slowly. The Roadrunner operated with exaggerated parameters—he accelerates very fast! These differences in acceleration reveal differences in the object’s character—differences that are crucial to animation. Objects in nature often begin to move in small increments that become larger until the object approaches a constant velocity; the increments might become smaller as the object decelerates to a stop.

Because acceleration is the rate of change of velocity per time, time enters the equation twice and it is therefore expressed in units such as miles/hour/hour. It may also be expressed in forms such as miles/hour/second, feet/second/second, feet/second², or mile × hours⁻². Animated objects that move in equal distances from the start to the finish of a motion do not accelerate and appear to start and stop with an unsightly jerk. Simply put, it is necessary to accelerate the object in order for it to appear real. Furthermore, because velocity is a measure of both speed and direction, any time there is a change in direction, even though the speed is constant, there must also be an acceleration. It is important to note that acceleration also pertains to actions that do not involve a change in position—for example, when lights are faded down, the decrease in brightness must accelerate.
Figure 10.7 Right: A blimp changes velocity slowly; a disaster changes velocity quickly. Acceleration occurs whenever an object speeds up or begins moving in a new direction.

Figure 10.8 Below: Linear acceleration. A vehicle beginning its travel at 0 miles/hour, and then uniformly accelerating 10 miles/hour for each of 5 hours, will eventually be traveling at 50 miles/hour.

Acceleration may be uniform or variable. Uniform acceleration is motion in which an object is accelerated at a constant rate; it is sometimes also called linear acceleration (see Figure 10.8). Uniform acceleration simulates the manner in which many objects in the real world gain speed. An object that moves 2 feet in the first second, 4 feet in the second second, and 6 feet in the third second is an example of uniform acceleration: it is gaining velocity, but the rate of acceleration is constant at 2 feet/second/second. It has been experimentally determined that the distance traveled during an inter-
val of time is equal to one-half the acceleration times the duration of
time squared, or in classic notation:

$$s = \frac{1}{2} at^2$$

Acceleration produces a parabolic curve when distance is plotted
against time. Note that although the rate of speed for linear accelera-
tion is constantly increasing, the rate of acceleration is constant.

It is also possible for an object to accelerate so that the rate of
acceleration is increasing. This is called variable acceleration. One case
of variable acceleration is exponential acceleration, when accelera-
tion is increasing at an exponential rate (see Figure 10.9). One example
of where this occurs in the real world is with a rocket; the rocket weight
primarily consists of its own fuel, so as it burns the fuel, the rocket
engine has less mass to push and is therefore able to propel faster.

Several pertinent equations involving velocity and acceleration
are as follows. The average velocity equation is only for the case
where velocity is increasing at a steady rate. The last equation
implies uniform acceleration.

velocity = acceleration × time
distance traversed = average velocity × time
average velocity = (initial velocity + final velocity)/2
position at time $t = \frac{1}{2} at^2$

**Derivatives**

Closely allied to acceleration is the concept of a derivative, especially
with regard to changes in the rate of motion. In this context, a
Figure 10.10 The first derivative of velocity is acceleration. The lower graph shows an object accelerating at a constant rate of 180 miles/hour for a duration of 2 hours. The top graph shows the same object with an initial airspeed of 120 miles/hour. Assuming the acceleration rate shown in the lower graph, the object in the upper graph will have an airspeed of 300 miles/hour at the end of the first hour and one of 480 miles/hour at the end of two hours.

*derivative* is a rate of change (see Figure 10.10). Thus, the first derivative of motion is the rate of change of the velocity, in other words, the acceleration. A constant-velocity motion is devoid of acceleration; it has no change in velocity and therefore its derivative is zero. The second derivative of velocity is the rate of change of the rate of change, or, in motion terms, any change in the rate of acceleration. When the value is non-zero, the result is called *jerk*; other synonyms might include *lurch* and other words to suggest an abrupt change in
speed, direction, or rate of curvature. Obviously, in a formal sense, the jerk may be zero, as is the second derivative of a constant-velocity motion. Note that the first derivative of velocity (acceleration) contains time twice, and that the second derivative of velocity (jerk) contains time three times, for example, miles/hour/hour/hour.

In the example of uniform linear acceleration (again see Figure 10.8), the change in velocity is 10 miles/hour/hour; thus, the first derivative is 10 miles/hour/hour and the second derivative, the change in acceleration, is zero. In the example of exponential acceleration (again see Figure 10.9), the velocity has both a first and a second derivative, because not only is the vehicle accelerating (changing velocity), but also the rate of acceleration is increasing. Here the first derivative of velocity is 5-10-15-20-25 miles/hour/hour and the jerk (the second derivative of velocity, or first derivative of acceleration) is a constant 5 miles/hour/hour/hour.

This process of rates of change of rates of change can continue indefinitely, and yes, there are third derivatives, and fourth derivatives, and so on. The reason that this matters is because humans can perceive change, and can perceive the rate of change. Hence, something that is changing will not appear smooth if the rate of change is not continuous; that is, the action will appear jerky if the rate of change varies. Examples of jerk and jerk-free action can be perceived while driving on clover-leaf exit ramps on roadways. On a well-designed ramp, the driver can turn the steering wheel into the turn and can increase the turn of the wheel at a continuous rate. On a poorly designed ramp, the driver must rotate the position of the steering wheel at various rates in order to move in the direction of the curve. Although both ramps curve (have change in direction), the well-designed ramp has a uniform rate of curvature, while the poorly designed ramp does not. Moreover, when the end of the ramp reconnects to a straight roadway, well-designed ramps connect in a continuous fashion.

Understanding derivatives is essential for creating smooth action, and in general, action must contain a first-derivative continuity in the temporal domain and a second-derivative continuity in the spatial domain in order to appear smooth. It is sometimes necessary to determine the derivatives of a sequence of values, and this is done by subtracting adjacent values (see Figure 10.11).
Motion Pathways

A motion pathway or motion curve is the route along which an object moves (see Figure 10.12). More specifically, it is a line or a curve along which an object moves through a sequence of frames. In mathematics, the term trajectory is used to mean the same thing, but the meaning is independent of media.

A motion pathway can be extremely simple, or it can be complex, such as a fractal pathway. Motion pathways can exist in 2D or 3D space. They are usually independent of whatever data describes the traveling object, and they are designed as an entity in their own right, independent of objects. Thus, many different objects can travel along the same motion pathway. This separation of motion from object is a feature of most computer animation systems. Finally, motion pathways can be used for the path of any object, including a camera or lights.

Motion Pathways in Practice

The simplest motion pathway is a straight line or a simple curve that describes a sequence of positions in space (see Figure 10.13). In Figure 10.13(a), the object moves along the path without any change to the object itself. Notice that the orientation of the object remains the same as the object traverses the entire motion pathway. A slightly more complex situation is presented in Figure 10.13(b), where the same object travels along the same pathway, but where the object rotates as it travels so that the object remains tangent to the curve at all times. In Figure 10.13(c), again the same pathway is used, but this time the object rotates as it travels, so that the object...
remains oriented on an external object. The approaches presented in Figures 10.13(b) and 10.13(c) are extremely useful when the object moving along the path is a camera. In Figure 10.13(b), the camera interest remains fixed on the pathway. In Figure 10.13(c), the camera interest remains fixed on an object.

A slightly less typical approach uses the same pathway from the previous example, but involves the object being scaled as it moves along the path (see Figure 10.13(d)).

In Figure 10.13(e), the same pathway is used again, but this time not only is it used to position the object, it is also used to deform the object as the object travels; this approach is called path deformation, also known as spline deformation and curve deformation. The curvature of the pathway itself is used to deform the shape of the object, at every corresponding position along the curve.

Finally, the pathway itself (whether simple or complex) can be extruded into a surface, such that the surface becomes a pathway; see Figures 10.13(f) and 10.13(g). In the figures, the pathway has a ribbonlike quality and twists and turns through 3D space; in mathematical terms, the surface normals along the path change orientation in xyz. Thus, the twists and turns of the pathway can be used to twist and turn the object as it moves along, as shown in the figure. In Figure 10.13(f), the object moves along the pathway, remaining tangent to it, but not deforming. In Figure 10.13(g), the object moves along the pathway and is deformed accordingly as it goes.

Any combination of the approaches listed above (translation, scale, rotation, and deformation) can be applied to any object as it moves along a pathway.

**Motion Pathway Vocabulary**

Some motion pathways are so common that they are given names. The idea of a vocabulary of motion pathways is hardly unique to computer animation. In many specialized fields—skiing, aviation, dance—compound motion pathways are important enough to be given names. For example, in aviation, which consists of free-form navigation of a 3D object in a 3D space, there are numerous standard motion pathways, which almost always involve translations (because the plane is flying) and rotations. Examples include a *loop* (a translation incorporating a 360-degree rotation about the y-axis), a *roll* (a

![Figure 10.13 Motion pathways.](image)

(a) the triangle does not rotate as it traverses the pathway. In (b) the triangle rotates to remain tangent to path. In (c) the triangle rotates to remain fixed on an external object. In (d) the triangle scales up and down as it moves along the path. In (e) the triangle deforms as it moves along the path. In (f) it rotates as it moves, but does not deform. In (g) it deforms as it moves along the path. Note that in (f) and (g), the path rotates in three dimensions (xyz), and thus the triangle rotates in xyz.)
Figure 10.14: Right: Lazy eight. This maneuver derives its name from the manner in which the airplane is made to trace the figure of an eight lying on its side. A lazy eight consists of two 180-degree turns, in opposite directions, while making a climb and a descent in a symmetrical pattern during each of the turns.

Figure 10.15: Above: Motion pathways based on a circle: circle, arc, ellipse, and parabola.

translation incorporating a 360-degree rotation about the x-axis), a spin (a translation incorporating a 360-rotation about the z-axis), a two-turn spin, a stall, and a lazy eight (see Figure 10.14).

The idea here is that motion pathways may often be thought of as entities with names, and directed accordingly. With respect to directing computer animation, one goal is to employ a high-level language and have the programs break down the action.

**Motion Pathway Primitives and Functions**

There are an infinite number of motion pathways, the most common of which are based on a circle (see Figure 10.15), including a circle itself, an arc (part of a circle), sine wave, ellipse, parabola, and hyperbola. Some of these are available as primitive curves in computer animation programs.

Beyond the circular functions, a rich collection of mathematical functions can be used to create pathways as well (see Figure 10.16). One of the reasons that functions are so useful to the animator is
that they are parameterized so that the incrementation of one or more input parameters produces a new position on the curve. A word of caution, however, is in order: equal increments of an input parameter to a function do not necessarily produce equally incremented points on the corresponding curve. Thus, to produce curves using functions, it is useful to know about the relationship between input parameters and the resulting curve.

**Motion Pathways and Splines**

Possibly the most versatile motion pathways are those created with splines. When using splines, just a few control points can be used to create and edit a complex curve. In addition, and possibly most importantly, splines can be used to create smooth, jerk-free motion pathways. Any of the standard splines can be used to create motion pathways. Acceleration and speed can be maintained as an object travels down the spline, as when a car drives over a road. An object can also be rotated so that it always stays normal to the pathway.
Straight-Ahead Animation

Straight-ahead animation is animation that is created by drawing the beginning frame of the animation, then the next frame, and animating continuously through until the end. The action has a beginning frame, but it is not bounded by an end frame, as is key frame animation. There is no predetermined destination; there is simply a flow of action. The first frame is drawn, then the second, then the third, and the action evolves, unfolding itself, as the frames progress.

In computer terms, straight-ahead animation may be produced kinematically, procedurally, dynamically, or behaviorally. It may certainly involve acceleration mechanisms as long as they are not predeterminant. It may involve a motion pathway and a destination, but it does not involve a definition of a destination time, for if it did, it would be a key frame approach, not a straight-ahead approach.

Key Frame Animation

The term key frame animation originated in the days of manual animation and defined a production process whereby the principal animator broke down all action into a series of key actions. In the traditional cel studio, the key action frames were drawn by the senior animator, whereas the in-between frames were drawn by an assistant. This specialization of labor is typical of industrial processes.

Key frame animation is an animation method where the action is bounded by a pair of key positions—also known as key frames, keys, and extremes—that represent the extent of action with a duration of time (see Figure 10.17). For example, the key positions of a clock pendulum would be the leftmost and rightmost positions of its swing. A more complicated example might be the key positions of a football player throwing a football: two key positions might be the player’s hand positioned behind the player’s head, holding the ball, and the player’s hand positioned in front of the player, at the moment of releasing the ball. A key is often typified by (1) a moment of rest and no acceleration, or (2) a moment of maximum position, zero velocity, and maximum acceleration.

Typically, the keys are created by the master animator. The frames between the keys are called in-betweens, and their images are formed by a breakdown of the information contained in the keys.
plus additional information about the nature of the object in motion. Appropriately, the process of making the in-betweens is called in-betweening or tweening. The in-betweener is a person (or computer program) who creates all the in-between drawings.

In computer animation, key frame animation can be created in both 2D and 3D systems. The 2D approach is often a direct simulation of the cel process whereby an animator draws the extremes and a computer calculates the intermediate shapes and positions; the goal here, to eliminate the human in-betweener, has proven more difficult than anticipated because often the in-between drawings are not simply a blend of the two extreme positions. When applied to 3D graphics, the key frame concept involves parameterizing the world, setting (either explicitly or interactively) the parameters of the extremes, letting the computer calculate the in-between parameters, and using these parameters to move (or deform) the object.

It is worth emphasizing that in key frame animation, the information for the start frame and the end frame is predetermined. The in-betweens can be calculated at any interval and in any order; there is no dynamic relationship between them; the present is not dependent upon the past. Thus, the conceptual difference between straight-ahead animation and key frame animation is that in key frame animation, the action is bounded by the extremes, and straight-ahead animation is not. The key frame animator animates from those extremes and sees the world as a sequence of gestures and movements that have beginnings and ends. The key frame animator knows that as they draw the 2nd drawing of a series, the 16th drawing of that series is already predetermined. The straight-ahead animator simply lets the action manifest itself, without a predetermined destination. Both techniques are useful approximations of the real world.

Key Charts and Breakdown Drawings
To assist in the in-between process, the animator often designs a key chart, which accompanies the first key frame (see Figure 10.18). The chart specifies where the key frames will fit into the sequence of an animation, and the length of the sequence in frames. In this example, there are five frames in the sequence; the odd number assures that there will be a middle frame. The chart also specifies which in-between frame is to be drawn first; this frame is called the breakdown drawing.

![Key Chart](image)
and is sometimes the frame exactly in the middle of two given key frames; in this example, the breakdown drawing is frame 3. The more in-betweens, the more frames it takes to present an action, hence, the slower the action. The key chart may also be used to designate changes in speed—for example, the chart can specify where (when) the in-betweens lie in the animation (see Figure 10.19). Remember, animation is not always a science, and like other art forms, can exaggerate, distort, and warp not only geometry, but also timing.

**Drawing Representation:**

**Discrete Cells Versus Action Parameters**

Key frames can be 2D, 3D, pixel- or voxel-based, or polygon-based. A key frame drawing may be a 2D outline drawing, be it a pencil line drawn by hand or a digital 3D polygon outline. It may be a sprite, that is, a pixel grid able to be moved over a background. It may also be a 3D vector object or voxel object. How the object is represented affects the key frame process.

An alternate approach is to not define the keys and in-betweens in terms of drawings, but rather in terms of parameters that define the shape and position of the object. In this approach, in-betweening involves the interpolation of the parameters from initial to terminal values. These parameters, or *action parameters*, are any parameters that define the object, including those that specify position, scale, rotation, and so on.

**Parametric Key Framing of Action Parameters**

*Parametric key frame animation* is animation that is represented by and created through the manipulation of action parameters, which change over time and which are defined by start and end values and a rate of change (see Figure 10.20). In the parametric key frame approach, the parameters are modified and the drawing is calculated; the start and end values are analogous to the start and end drawings.

Three-dimensional computer animation that is accomplished using parametric key frame animation involves the following steps:

First, the key objects are created and positioned. Then, the in-between values of the action parameters are automatically calculated by the animation software and applied to the object, creating the in-between drawings.
In 3D computer animation, everything in an animation set is parameterized—for example, the position of objects, lights, and camera; the color of lights and objects; textures and reflections; degrees of transparency; density of atmospheric haze; and so on. The computer animator's art lies in the manipulation of these parameters on a frame-to-frame basis. Just as objects and lighting require design, so, too, must action be sculpted in space.

Keep in mind that although each controllable degree of freedom in a model or animation can be assigned a parameter, there are many ways to define key values. Obviously, they can be defined interactively, by creating an object and then positioning it. But they can also be defined explicitly, by typing in values at the keyboard. And an object's shape and position can be derived from the real world, using rotoscoping or even sophisticated 3D motion tracking.

The parametric approach to key frame animation allows at least two ways of achieving actions: (1) by parameterizing the object and interpolating from an initial object to a terminal object, changing its geometry (see Figure 10.21), and (2) by parameterizing an action transformation, which gets applied to the object; the parameters being interpolated are usually those of the three axes transformations—translation, rotation, and scale. The translation transformation moves objects left, right, up, down, in, or out (see Figure 10.22). The scale transformation makes the object bigger or smaller (see Figure 10.23). And three-rotation transformation pivots an object around one of three axes (see Figure 10.24). Transformations can be combined, or concatenated, creating one transformation that combines the successive order of two or more primitive transformations. For example, there may be one transformation that can perform a rotation followed by a scale.

Do note that parametric methods in general are widely used in computer animation, and that many parametric methods (inverse kinematics, dynamics, procedural) do not require key framing. In the key frame system, although the model may be fully parameterized, it is always constrained by the extremes.

**Easing**

It is important to understand that the quest for reality is only partly in how objects look; reality must also involve believable action. Animators
need to understand how action really occurs and how to approximate it. The focus of this section is on easing methods that achieve believable action, albeit not necessarily natural action. This section examines both ad hoc methods as well as eases that mimic nature.

Easing and key framing are both about action and, hence, easily bound together, but are definitely not the same thing. Key frame animation requires a start state of the action, an end state of the action, a start time, an end time, and a rule, called an ease, or easing, which specifies how the in-betweens are calculated. For example, an ease
would determine whether an action will begin slowly and end quickly, or begin quickly and end slowly. Eases also provide a way to depict acceleration and deceleration. In other words, eases are functions that calculate rates of change.

If, for example, the key frames define a start position and an end position of an object, and if there is a defined motion pathway, then an ease will dictate the individual positions of the object for each frame as it moves along the pathway. Most natural action starts gradually, and then increases speed. When an ease starts off slowly, it is said to ease in. Conversely, when an ease ends slowly, it is said to ease out.

In formal terms, easing is a 1D time-to-curve mapping; it maps a point in time to a point on a curve. There are two ways to define an ease: (1) it can be defined so that it returns a set of solutions for a given number of frames (times), or (2) it can be defined to return a solution for a given percentage value, where the percentage value represents a percent of the time along the action. In both cases, the function takes even divisions of time and returns uneven divisions of action.

Eases are used in all approaches to animation, whether hand drawn, motion graphics, or computer-generated. Eases are used to control the movements of objects, cameras, zooms, focus, color and lighting changes, and special effects such as fades, dissolves, and glows (see Figure 10.25). Practically speaking, everything that changes, eases. A camera that is meant to move, eases; the camera must be pushed, and it has to gain momentum and speed. Similarly

![Figure 10.25](image-url) - An easing world. The dimmer switch at the left can be eased up and down; thus, the spotlight at the top can be eased on and off. In the middle of the scene is a camera on a dolly, which can be eased up and down, and in and out; its rotation can also be eased. At the right is a balloon, which can be eased up and down.
for zooms, the zoom ring has to be turned and eased. Lights fading in or out, or changing color—all have to be eased. Morphing—changing from one image to another—requires easing in and easing out on the control points. Any transition effect must be eased.

In a practical sense, the only time one does not need to ease in or ease out is when an object—a moving car, a meteor—enters into the frame already at full velocity; the object is moving at a constant velocity because it must have already reached constant velocity before it entered the frame. Hence, not everything moving in the field of view is in a moment of easing in or out. A camera panning a scene may already be at rotational velocity before there is a cut to the scene. But if a cut to the scene is made before the camera starts moving, it must ease-in when it first starts to move. It is important to note that a scene starting with the camera at rest followed by an ease-in has a different feeling than a scene in which the camera is already in motion.

It is important to emphasize that when the ease being performed is with respect to a change in position, the ease assumes that the pathway along which the object is moving is already defined.

The creation of eases is quite open-ended; some produce better results than others.

**Ease Diagrams**

An ease diagram, also known as a timing curve or function curve, is a plotted curve with time along the x-axis and the parameter being eased along the y-axis (see Figure 10.26). In Figure 10.27, the car accelerates slowly (eases in), as shown in the left part of the ease diagram. The car then moves at a steady speed, as indicated by the straight, middle part of the curve. And finally, the car decelerates to a stop (eases out) as indicated by the right part of the diagram. In computer animation, the ease diagram might be automatically created when an action is created, or it can be drawn by hand. Every action has a corresponding ease, and every ease has a unique ease diagram.
It is important to note that timing curves are entities in themselves and exist independently of an action. They can be saved, retrieved, translated, scaled, reversed, cut, copied, and pasted (see Figure 10.28). Points can be added to or deleted from the curve, and points on the curve can be edited as well. Each change applied to the curve alters the action that might be associated with the ease. Finally, an ease diagram can be applied to any number of different actions.

**Issues of Continuity**

Easing is a rate-of-change problem. If an object is motionless, for example, and then is suddenly in motion, the rate of change has been discontinuous, so the action will jerk. If an object is traveling at a constant 60 miles/hour one moment and the next moment is traveling at a constant 70 miles/hour, even though both speeds may have been constant, the rate of change of speed is discontinuous, and thus the motion will jerk. In order for the action to appear natural and continuous, the rate of change of action must be constant. Actions can change speeds, but they must change speeds at a constant rate of change to appear natural. In cases where the acceleration is not consistent, then the rate of change of the acceleration must be consistent.

**Types of Eases**

An action can ease in an infinite number of ways. It can begin quickly or slowly, or it can proceed at a constant rate, or it can happen at a random rate. This section presents types of commonly used eases. The reader is cautioned that not all of these eases are designed to produce a smooth continuity of action, and that our definition of ease is that it is a rule to define a continuity of action, and that a rule to mimic smooth action is but one case.

**Linear Interpolation Ease**  
*Interpolation* is a method used to calculate intermediate values that are between two existing values, where the quantity of in-between values is specified, and where the values can represent anything from position to shape, time, or behavior. Interpolation is a technique widely employed in computer graphics for a variety of purposes, including animation.

![Figure 10.28 Editing timing curves. In (a) the curve is translated; translating the ease curve is the same as translating the action in time—a curve that is translated to the right causes the action to happen later. In (b) the curve is scaled larger in x. In (c) the curve is scaled smaller in x. Scaling an ease curve is the same as making an animation longer or shorter—when a curve is scaled smaller, the animation is made shorter; when a curve is scaled larger, the animation is made longer. In (d) the curve is copied and reversed. In (e) it is copied and repeated. In (f) a point is added to the curve. In (g), by moving the new point, the curve is modified. In (h), given a curve with several points, one point is deleted.](image-url)
Linear interpolation is an interpolation in which the calculated values are equally spaced between the two existing values (see Figure 10.29). Linear interpolation is perhaps the simplest way of getting from one state to another in a designated amount of steps, and hence is used to create easing, in particular, linear interpolation eases, or linear eases (see Figure 10.30). The parameter being eased can represent anything that can be parameterized—for example, a distance along a curve or an angle around an arc. In a linear ease of distance along a curve, an object would increase its distance toward the end position in equal increments for each key frame. In a linear ease of an angle around an arc, an object would increase its angle toward the end angle in equal increments for each key frame.

While this solution is valid, it does not produce natural-looking results because it does not simulate acceleration or deceleration as manifested in nature. That is, most objects in the real world do not move in a linear fashion—for example, they decelerate to a stop. A linear ease produces a jerk in the action (a second derivative discontinuity) at the start and end of the shot and are not acceptable for most cases. This does not mean that it is without uses. It is fine for handling action that is passing through the screen, assuming that one is viewing an object already in motion. It is often used by scientific visualizers for whom even steps are often more important than those that mimic Newtonian physics (involving acceleration and mass).

Anything that can be represented numerically—position, scale, angle of rotation, color, and surface normals—can be interpolated. All that is needed is the initial value, the terminal value, and the number of in-between values that are to be calculated. Figure 10.29 presents a simple example of linear interpolation, suppose that there are two points on a plane, the initial point (3, 4) and the terminal point (7, 6). And suppose that three in-between points are needed, totaling five points. A linear interpolation is calculated by subtracting the initial point values from the terminal point values, creating a total x displacement value and a total y displacement value, in this case 4 and 2. Each in-between x and y displacement value is equal to the total displacement value divided by the total number of interpolation points minus 1 (in this case 5 – 1, or 4). Thus, the x displacement value for each in-between is ⅕ or 1, and the y displacement value for each in between is ⅕ or 0.5. These displacement values are then succe-
sively added to the original point values, to create a sequence of linear interpolated positions:

3. 4 initial point; 4, 4.5; 5; 6, 6.5; 7, 6 terminal point.

Linear interpolation can be described in more formal terms as such. Given two points, \( P_1 \) and \( P_2 \), and \( t \), where \( t \) is time relative to the two points and \( 0 \leq t \leq 1 \), any in-between point can be calculated according to the following formula:

\[
P(t) = (1 - t)P_1 + tP_2
\]

The equation \( P(t) = (1 - t)P_1 + tP_2 \) is the parametric definition of a line, used to calculate the in-between values for linear interpolation. Therefore, in the previous example, the midpoint—\( t = \frac{1}{2} \)—could be calculated as:

\[
P(\frac{1}{2}) = (1 - \frac{1}{2})P_1 + (\frac{1}{2})P_2
\]

For \( x \): \( P(\frac{1}{2}) = (1 - \frac{1}{2})3 + (\frac{1}{2})7 = (\frac{1}{2})3 + 7 = 5 \)

For \( y \): \( P(\frac{1}{2}) = (1 - \frac{1}{2})4 + (\frac{1}{2})6 = (\frac{1}{2})4 + 6 = 5 \).

Other kinds of eases can be more complex, such as eases that change according to sine waves or logarithmic progressions, or those that are calculated using the mathematical formula for acceleration, so that actions occur in a natural way.

**Classic Breakdown Ease** The classic breakdown ease is an ease that is created by recursively subdividing from the center of the action outward toward the two extremes. That is, the percentage of the parameter being eased is first divided into half, and then the outer halves are divided into halves, and then the outermost quarters are divided into half, and so on (see Figure 10.31). Action is often drawn on an odd number of cells so that there shall be a middle frame. This subdivision process produces a multiframe sequence with a slow ease-in, rapid velocity in the middle of the action, and a slow ease-out. Classic breakdown eases are simple to create and the results resemble natural acceleration and deceleration, but they are not mathematically accurate. Applied recursively, subdivision is a basis of thousands of computer algorithms, including rendering, compression, hidden surface removal, and anti-aliasing.
Linear Acceleration Ease A commonly used ease, certainly whenever a computer is being used to calculate it accurately, is the \textit{linear acceleration ease} (see Figure 10.32), which uses the formulas of velocity, acceleration, and deceleration (presented earlier in this chapter) to ease parameters. That is, these formulas are applied to a parameter, such as an angle, and the parameter is eased accordingly. The parameter can be eased either indefinitely, or until a desired velocity is reached (when accelerating), or until the action achieves a state of rest (when decelerating). The linear acceleration ease diagram illustrates a smooth transition from zero velocity, and hence is useful for animating an action that starts at a rest state. It can also be reversed to animate an action coming to a stop.

Linear acceleration simulates the way action behaves in the real world. When an action begins, it usually begins in small increments, which become increasingly larger as the action approaches a constant speed. For example, when a camera starts to dolly, it begins to dolly slowly, until it achieves the desired speed. When an action ends, it becomes progressively slower as the action slows to a stop— as an ordinary automobile might gradually slow down, eventually braking to a stop.

Remember that when in-betweening, the origin and destination are known; therefore an ease must calculate the acceleration based on the change incurred during the ease-in. The deceleration is based on the change incurred during the ease-out. And the \textit{coast}, or constant velocity portion of the ease, is based on dividing the remaining change in action into equal increments. We must solve for the velocity and acceleration parameters so that the acceleration, coast, and deceleration all work together such that the in-between values lie between the two extremes.

\textbf{Sine Wave Ease} Sine waves provide a convenient way to create action, especially action that is oscillating (see Figure 10.33). A \textit{sine wave ease} is an ease derived from a circle and can be used to approximate many biological and natural manifestations, especially for objects in a steady state of motion—a dog’s tail wagging, a person breathing, ocean waves, swaying trees, a walking person, a cork bobbing on water, and certain types of machinery such as pistons connected to rotating shafts.
A sine wave ease is most applicable for action that oscillates between two extreme positions. The action produced by a sine wave ease begins slowly, achieves a fairly constant velocity, and then slows to a stop. For example, a little girl might be holding a balloon that is bobbing up and down in a smooth fashion. The balloon moves along a straight line (the assumed pathway) between a lower point and a maximum height (see Figure 10.34).

The action based on a sine wave ease is markedly similar to the action based on a linear acceleration or deceleration ease, but it is not the same. The sine wave motion starts to slow a little sooner, and it has a more rounded toe and shoulder that creates a more organic feel to the motion.

Figure 10.33 Left: Sine wave ease. The sine wave is created from a function that uses an angle as a parameter; the angle has a range of \(-90\) to \(+90\) degrees, domain of the sine is \(-1 \leq \sin \leq 1\).

Figure 10.34 Above: A balloon bobbing up and down on a straight line according to a sine ease. The action produced by a sine wave ease begins slowly, achieves a fairly constant velocity, slows to a stop, then reverses direction.
The sine function is attractive because it is centered at zero, and it has a domain of $-1$ to $+1$. Hence, it is ideal for doing oscillatory motion around a center (although this can be normalized to a $0$ to $1$ domain). But because it has an initial and terminal speed of zero, it can also be used to ease between two extremes.

**Random Ease** While it is often the case that eases impose some logical, orderly progression on an action, this does not necessarily have to be the case. An ease can be completely random—that is, a *random ease* (see Figure 10.35). For example, suppose that the scene is a room that includes a jack-o'-lantern; the glow of the lantern fills the room with randomly flickering light. To animate the light as seen inside the room, the change in the lighting might be extremely fast at first, followed by very slow, followed by another fast change, and then an even faster change. This sort of random ease might be necessary in order to achieve a particular emotional impact. Because computers are very good at computing random numbers, computers are also good at creating random eases, and any action parameter being eased can be eased randomly.

**Empirical Ease** Eases can also be designed empirically. An *empirical ease* can be designed by observing natural action (see Figure 10.36). For example, if the action in a scene is to simulate the throwing of a ball, then an actual pitch can be photographed and then
rotoscoped to determine the position of the ball in each frame. The rotoscopied positions of the ball may then be digitized and analyzed to create an ease for an animated ball; they can also be used to create a motion pathway of a real ball.

**Spline Ease** Eases can also be created by hand; the animator draws the ease diagram or the points that compose it. In today’s world, it is more efficient to draw only a few points and employ the computer to construct a spline—the spline may or may not pass through the points, depending upon the type of spline used.

The beauty of this method is that a spline allows for the ease to progress smoothly. This is because the spline not only curves smoothly, allowing for continuous acceleration and deceleration, but it also progresses smoothly between different rates of acceleration and deceleration. This means that action easing on a spline will not jerk, which is often a critical requirement. Any of the various spline forms, such as Bezier splines or B-splines, can be used. This method allows one to create action that might move and rest—consider the flight of a hummingbird, for example.

**Subtleties in Easing—Anticipation and Overshoot**

*Anticipation* is when a large action is preceded by a smaller action, in the opposite direction, that sets up, or anticipates, the larger action (see Figure 10.37). For example, before moving forward, the motion first goes back a short distance and then proceeds. More precisely, when throwing a ball forward, a person pulls their body back first, and then moves forward. Conversely, *overshoot* is when a large action is followed by a smaller action in the reverse direction. For example, a forward motion comes to a rest by rocking a bit, as one might do on a train when it is coming to a halt (see Figure 10.38). Such subtleties can be built into eases (see Figure 10.39).

**An Advantage of Easing**

In an animation, some frames are more important than others—they represent a point in time that is critical to the action or story. Thus, sometimes, animators need to look at one frame in an animation before the entire animation is made. When easing during key framing, the in-between frames can be solved in any order. For example, easing
enables the animator to calculate the position of an object at a frame 60% into the ease, without creating all of the preceding frames first. It is also possible to calculate all of the positions for all of the objects in the frames before creating the frames, which is a common practice. Once all of the parameters are calculated, the parameters can be stored and used to create the frames; if any particular frame needs to be re-rendered, the data for that frame will be available.

**A Final Word on Easing and Key Frame Animation**

It is important to note that easing techniques need not necessarily be bound to key framing. For example, an object can be made to accelerate without any predefined terminal position. Also, in-betweening is not the only way to do realistic motion with 3D parametric objects. For example, motion can be calculated dynamically—a rocket ship could be launched into space, where the rocket has a simulated rocket motor and a steering mechanism and the Earth has simulated gravity that is exerted on the mass of the rocket. As the rocket is launched into the dynamically modeled atmosphere, the animation that is created is a result of the forces of the environment. These dynamic approaches are discussed in greater detail in Chapter 12.